

# Homogenization of reflected semilinear PDE with nonlinear Neumann boundary condition.

Auguste Aman\* and Modeste N'Zi†  
 UFR de Mathématiques et Informatique,  
 22 BP 582 Abidjan 22, Côte d'Ivoire

## Abstract

We study the homogenization problem of semi linear reflected partial differential equations (reflected PDEs for short) with nonlinear Neumann conditions. The non-linear term is a function of the solution but not of its gradient. The proof are fully probabilistic and uses weak convergence of associated reflected generalized backward differential stochastic equations (reflected GBSDEs in short).

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**Key Words:** Reflected backward stochastic differential equations; homogenization of PDEs; viscosity solution of PDEs; obstacle problem.

## 1 Introduction

Let  $L_x^\varepsilon$  be an uniformly elliptic second-order partial differential operator indexed by a parameter  $\varepsilon > 0$ . The homogenization problem consists in analyzing the behavior as  $\varepsilon \downarrow 0$ , of the solution  $u_\varepsilon$  for the partial differential equations (PDEs):  $L_x^\varepsilon u_\varepsilon = f$  in a domain  $\Theta$  of  $\mathbb{R}^d$ . The coefficients of the operator  $L_x^\varepsilon$  are subject to an appropriate conditions. For example, periodicity by Freidlin [5], almost periodicity by Krylov [9]. In this paper we shall consider the case of locally periodic coefficients introduced by Ouknine et al. [13]. In the probabilistic approach the problem becomes the following. What is the limit of the laws of the diffusion processes  $X^\varepsilon$  with generator  $L_x^\varepsilon$ , as  $\varepsilon \downarrow 0$ ? This problem has been studied for diffusion processes in the whole of  $\mathbb{R}^d$  by Krylov[9], Freidlin[5] and Bensoussan[1]. In the case of the presence of

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\*E-mail address: augusteaman5@yahoo.fr (Corresponding author)

†E-mail address: modestenzi@yahoo.fr

boundary conditions, we can see the work of Tanaka [18] and recently this one of Ouksine et al.[13]. In [15], Pardoux has combined the probabilistic approach of chapter 3 of [1] to derive homogenization results for semi linear parabolic partial differential equation involving a second order differential operator of parabolic type in the whole space with highly oscillating drift and nonlinear term. Furthermore Pardoux and Ouksine [12] pursue the same program for a semi-linear elliptic PDE with nonlinear Neumann boundary conditions.

This paper is devoted to prove a similar result for obstacle problem of semi-linear elliptic partial differential equations (PDEs in short) with a nonlinear Neumann boundary conditions  $g$  satisfying moreover linear growth conditions. We use for this purpose the link between backward stochastic differential equations (BSDEs in short) and PDEs.

The concrete formulation of BSDEs in the nonlinear case was first introduced by Pardoux and Peng [14] who proved existence and uniqueness of adapted solutions for these equations, under suitable conditions on the coefficient and the terminal value. They provide probabilistic formulas for solutions of semi linear PDEs which generalized the well-known Feynman-Kac formula. The interest of this kind of stochastic equations has increased steadily, since it has been widely recognized that they provide a useful framework for the formulation of many problems in mathematical finance (see [6]), in stochastic control and differential games (see [7]). A class of BSDEs called generalized BSDEs has been introduced by Pardoux and Zhang in [16]. This kind of BSDEs has an additional integral with respect to a continuous increasing process and is used to provide probabilistic formulas for solutions of semi linear PDEs with nonlinear Neumann boundary conditions. Further, in order to provide probabilistic formula for solutions of obstacle problem for PDEs with nonlinear Neumann boundary conditions, Ren and Xia [17] considered reflected generalized BSDEs. Our goal in this paper is to give a homogenization result of obstacle problem for semi-linear PDEs with nonlinear Neumann boundary conditions via a weak convergence of reflected generalized BSDEs.

To describe our result more precisely, we first recall some classical notations that will be used in the sequel of this paper. Let  $\mathcal{C}_b^1(\mathbb{R}^d)$  be the space of real continuously differentiable functions such that the derivatives are bounded and  $\Theta$  be a regular convex and bounded domain in  $\mathbb{R}^d$ . We introduced a function  $\rho \in \mathcal{C}_b^1(\mathbb{R}^d)$  such that  $\rho = 0$  in  $\bar{\Theta}$ ,  $\rho > 0$  in  $\mathbb{R}^d \setminus \bar{\Theta}$  and  $\rho(x) = (d(x, \bar{\Theta}))^2$  in the neighborhood of  $\bar{\Theta}$ . Let us note that, since the domain  $\Theta$  is smooth, it is possible to consider an extension  $\psi \in \mathcal{C}_b^2(\mathbb{R}^d)$  of the function  $d(x, \partial\Theta)$  defined on the restriction to  $\Theta$  of the neighborhood of  $\partial\Theta$  such that  $\Theta$  and  $\partial\Theta$  are characterized by

$$\Theta = \{x \in \mathbb{R}^d : \psi(x) > 0\} \quad \text{and} \quad \partial\Theta = \{x \in \mathbb{R}^d : \psi(x) = 0\}$$

and for all  $x \in \partial\Theta$ ,  $\nabla\psi(x)$  coincide with the unit normal pointing toward the interior of  $\Theta$  (see for example [10], Remark 3.1). In particular we may and do choose  $\rho$  and  $\psi$  such that  $\langle \nabla\psi(x), \delta(x) \rangle \leq 0$ , for all  $x \in \mathbb{R}^d$ , where  $\delta(x) = \nabla\rho(x)$  which we call the penalization term. Now let us consider the following reflected semi-linear partial differential equations with Neumann boundary condition for  $\varepsilon > 0$ ):

$$\begin{cases} \min(u^\varepsilon(t, x) - h(t, x), \\ -\frac{\partial u^\varepsilon}{\partial t}(t, x) - L_x^\varepsilon u^\varepsilon - f(x, u^\varepsilon(t, x))) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \Theta, \\ \Gamma_x^\varepsilon u^\varepsilon(t, x) + g(x, u^\varepsilon(t, x)) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \partial\Theta, \\ u^\varepsilon(0, x) = l(x), \quad x \in \overline{\Theta}, \end{cases}$$

where  $\Gamma_x^\varepsilon$  is defined in (2.2). Under suitable linear growth, monotonic and Lipschitz conditions on the coefficients  $f$ ,  $g$  and  $l$ , we will show that  $u^\varepsilon(t, x)$  the viscosity solution of the previous reflected PDEs converge, as  $\varepsilon$  goes to 0, to a function  $u(t, x)$ , the viscosity solution to the following reflected PDEs with nonlinear Neumann boundary conditions:

$$\begin{cases} \min(u(t, x) - h(t, x), \\ -\frac{\partial u}{\partial t}(t, x) - L_x^0 u - f(x, u(t, x))) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \Theta, \\ \Gamma_x^0 u(t, x) + g(x, u(t, x)) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \partial\Theta, \\ u(0, x) = l(x), \quad x \in \overline{\Theta}, \end{cases}$$

where operators  $L_x^0$  and  $\Gamma_x^0$  are in the form (2.4).

The rest of this paper is organized as follows. In section 2 we recall without proof, some results due to Ouknine et al.[13] on the homogenization of the reflected diffusion in the bounded domain with locally periodic coefficients and some basic notations on generalized BSDEs. Section 3 is devoted to the statement and proof of our main result. Finally, in Section 4, we give a homogenization result for some PDEs with nonlinear Neumann boundary conditions.

## 2 Locally periodic homogenization of reflected diffusion

In this section, we state some results which were proved in Ouknine et al. [13], and which we will use in the rest of the paper. Let us recall that  $\Theta$  is a regular convex and bounded domain in  $\mathbb{R}^d$ ; ( $d \geq 1$ ) defined as previous and let

$$L_x^\varepsilon = \sum_{i,j}^d a_{ij}(x, \frac{x}{\varepsilon}) \partial_i \partial_j + \frac{1}{\varepsilon} \sum_{i=1}^d b_i(x, \frac{x}{\varepsilon}) \partial_i + \sum_{i=1}^d c_i(x, \frac{x}{\varepsilon}) \partial_i \quad (2.1)$$

and

$$\Gamma_x^\varepsilon = \sum_{i=1}^d \partial_i \psi(x, \frac{x}{\varepsilon}) \partial_i \quad (2.2)$$

be given (with  $\varepsilon > 0$ ), where  $\partial_i = \partial/\partial x_i$ . We also set

$$L_{x,y} = \sum_{i,j}^d a_{ij}(x,y) \partial_i \partial_j + \sum_{i=1}^d b_i(x,y) \partial_i, \quad x, y \in \mathbb{R}^d$$

and require the following:

**(H.1)**  $L_{x,y}$  is uniformly elliptic and the matrix  $a(x) = [a_{i,j}]$  is factored as  $\sigma(x,y)\sigma^*(x,y)/2$

**(H.2)** The functions  $\sigma : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$ ,  $b : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$ ,  $c : \mathbb{R}^d \times \Theta \rightarrow \mathbb{R}^d$  are locally periodic (i.e, periodic with respect to second variable; of period 1 in each direction in  $\Theta$ ).

$$\left\{ \begin{array}{l} (i) \quad \text{Global Lipschitz condition: there exists a constant } C \text{ such that for any } \zeta = \sigma, b, c, \\ \|\zeta(x, y) - \zeta(x', y')\| \leq C (\|x - x'\| + \|y - y'\|), \quad \forall x, x' \in \mathbb{R}^d, y, y' \in \Theta, \\ (ii) \quad \text{The partial derivatives } \partial_x \zeta(x, y) \text{ as well as the mixed derivatives } \partial_{xy}^2 \zeta(x, y) \\ \text{exist and are continuous, } \forall x \in \mathbb{R}^d, y \in \Theta \text{ for } \zeta = a, b, c \\ (iii) \quad \text{The coefficients are bounded, that is, there exists a constant } C \text{ such that for any} \\ \zeta = a, b, c \quad \|\zeta(x, y)\| \leq C, \quad x \in \mathbb{R}^d, y \in \Theta. \end{array} \right.$$

The differential operator  $L_x^\varepsilon$  inside  $\Theta$  together with the boundary condition  $\Gamma_x^\varepsilon u = 0$  on  $\partial\Theta$  determine a unique diffusion process  $X^\varepsilon$  in  $\bar{\Theta}$  which we call the  $(L_x^\varepsilon, \Gamma_x^\varepsilon)$ -diffusion. Given a  $d$ -dimensional Brownian motion  $(W_t : t \geq 0)$  defined on a complete probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ , let  $(X^\varepsilon, G^\varepsilon)$  be the unique solution with values in  $\Theta \times \mathbb{R}_+$  of the following reflected SDE:

$$\left\{ \begin{array}{l} dX_t^\varepsilon = \frac{1}{\varepsilon} b(X_t^\varepsilon, \frac{X_t^\varepsilon}{\varepsilon}) dt + c(X_t^\varepsilon, \frac{X_t^\varepsilon}{\varepsilon}) dt + \sigma(X_t^\varepsilon, \frac{X_t^\varepsilon}{\varepsilon}) dW_t + \nabla \psi(X_t^\varepsilon) dG_t^\varepsilon, \quad t \geq 0 \\ X_t^\varepsilon \in \Theta, \quad G^\varepsilon \text{ is continuous and increasing such that,} \\ \int_0^t \nabla \psi(X_s^\varepsilon) \mathbf{1}_{\{X_s^\varepsilon \in \Theta\}} dG_s^\varepsilon = 0, \quad t \geq 0 \\ X_0^\varepsilon = x. \end{array} \right. \quad (2.3)$$

By requirement there exists a  $L_{x,y}$ -diffusion on  $\mathbb{R}^d$  with generator  $L_{x,y}$  and by  $y$ -periodicity assumption on the coefficients this process induces diffusion process  $U^x$  on the  $d$ -dimensional torus  $\mathbf{T}^d$ , moreover this diffusion process is ergodic. We denote by  $m(x, \cdot)$  its unique invariant measure. In order for the process with generator  $L_x^\varepsilon$  to have a limit in law as  $\varepsilon \downarrow 0$ , we need the following condition to be in force.

**(H.3) Centering condition:** for all  $x$ ,

$$\int_{\Theta} b(x, u) m(x, du) = 0.$$

Let us denote

$$\hat{b} = \begin{pmatrix} \hat{b}_1 \\ \hat{b}_2 \\ \vdots \\ \vdots \\ \hat{b}_d \end{pmatrix}$$

with  $\hat{b}^k(x, u) = \int_0^\infty \mathbb{E}_u \{b^k(x, U_t^x)\} dt$ , where under  $\mathbb{P}_u$ ,  $U^x$  starts from  $u$ .

We put

$$L_x^0 = \sum_{i,j=1}^d \bar{A}_0^{ij}(x) \partial_{x_i x_j} + \sum_{i=1}^d \bar{C}_0^i(x) \partial_{x_i}, \quad \Gamma_x^0 = \sum_{i=1}^d \gamma_i^0(x) \partial_{x_i}, \quad (2.4)$$

where the coefficients are respectively defined by

$$C_0(x, y) = \left[ \partial_x \hat{b} \hat{b} + (I + \partial_y \hat{b}) c + \frac{1}{2} \text{Tr} \partial_{xy}^2 \hat{b} \sigma \sigma^* \right] (x, y),$$

$$S_0(x, y) = \left[ (I + \partial_y \hat{b}) \sigma \right], \quad A_0(x, y) = S_0 S_0^*(x, y),$$

$$\bar{C}_0(x) = \int_{\mathbf{T}^d} C_0(x, u) m(x, du), \quad \bar{A}_0(x) = \int_{\mathbf{T}^d} A_0(x, u) m(x, du),$$

$$\gamma^0(x) = \int_{\mathbf{T}^{d-1}} \left[ (I + \partial_y \hat{b}) \nabla \psi \right] (x, y) d\tilde{m}(x, dy).$$

Let us consider the following SDE:

$$\begin{cases} dX_t = \bar{A}_0^{1/2}(X_t) dW_t + \bar{C}_0(X_t) dt + \gamma^0(X_t) dG_t, \quad t \geq 0, \\ X \in \Theta, \quad G \text{ is continuous and increasing}, \\ \int_0^t \gamma^0(X_s) \mathbf{1}_{\{X_s \in \Theta\}} dG_s = 0, \quad t \geq 0, \\ X_0 = 0. \end{cases}$$

The operators  $L^0$  and  $\Gamma^0$  are acting on  $x$ .

We can now state the main result of Ouknine et al. [13] (see Theorem 3.1), which is a generalization of a result of Tanaka [18] (see Theorem 2.2).

**Theorem 2.1** *Under assumptions **(H1)**-**(H3)**, the  $(L_x^\varepsilon, \Gamma_x^\varepsilon)$ -diffusion process  $X^\varepsilon$  converges in law to an  $(L_x^0, \Gamma_x^0)$ -diffusion  $X$  as  $\varepsilon \rightarrow 0$ .*

Moreover,

$$(X^\varepsilon, M^{X^\varepsilon}, G^\varepsilon) \Longrightarrow (X, M^X, G),$$

where  $M^X$  (resp.,  $M^{X^\varepsilon}$ ) is the martingale part of  $X$  (resp.  $X^\varepsilon$ ),  $G$  (resp.,  $G^\varepsilon$ ) is the local time of  $\psi(X)$  (resp.,  $\psi(X^\varepsilon)$ ) at 0 and

" $\Longrightarrow$ " designates the weak convergence in the space  $\mathcal{C}([0, T], \mathbb{R}^d) \times \mathcal{D}([0, T], \mathbb{R}^d) \times \mathcal{C}([0, T], \mathbb{R}^d)$  endowed with the product topology of the uniform norm for the space  $\mathcal{C}([0, T], \mathbb{R}^d)$  and the  $S$ -topology for  $\mathcal{D}([0, T], \mathbb{R}^d)$ .

It follows moreover easily from the result of Tanaka [18] that

**Lemma 2.1** *Under assumptions **(H1)** – **(H3)**,*

$$\sup_\varepsilon \mathbb{E} \left( \sup_{0 \leq s \leq t} |X_s^\varepsilon|^p + G_t^\varepsilon \right) < \infty,$$

for any  $p \geq 1$ .

### 3 Reflected generalized BSDEs and weak convergence

#### 3.1 Preliminaries and statement of the problem

Let us introduce some spaces:

$$H^{2,d}(0, t) = \left\{ \{\psi_s; 0 \leq s \leq t\} : \mathbb{R}^d\text{-valued predictable process} \right. \\ \left. \text{such that } \mathbb{E} \int_0^t |\psi(s)|^2 ds < \infty \right\},$$

$$S^2(0, t) = \left\{ \{\psi_s; 0 \leq s \leq t\} : \text{real-valued, progressively measurable process} \right. \\ \left. \text{such that } \mathbb{E} \left[ \sup_{0 \leq s \leq t} |\psi(s)|^2 \right] < \infty \right\},$$

$$A^2(0, t) = \left\{ \{K_s; 0 \leq s \leq t\} : \text{real-valued adapted, continuous, ,} \right. \\ \left. \text{and increasing process such that } K(0) = 0, \mathbb{E}|K(t)|^2 < \infty \right\}.$$

In addition, we give the following assumptions:

Let  $f, g : \bar{\Theta} \times \mathbb{R} \rightarrow \mathbb{R}$  be a continuous function satisfying the following assumptions: There exist constants  $C > 0$ ,  $p \geq 1$ ,  $\mu \in \mathbb{R}$  and  $\beta < 0$ , such that for all  $x \in \bar{\Theta}$ ,  $y, y' \in \mathbb{R}$ ,

$$(H.4) \left\{ \begin{array}{ll} (f.i) & (y - y') (f(x, y) - f(x, y')) \leq \mu |y - y'|^2, \\ (f.ii) & |f(x, y)| \leq C(|x|^p + |y|^2), \\ (g.i) & (y - y') (g(x, y) - g(x, y')) \leq \beta |y - y'|^2, \\ (g.ii) & |g(x, y)| \leq C(|x|^p + |y|^2). \end{array} \right.$$

Moreover, let  $l : \Theta \rightarrow \mathbb{R}$  be a continuous function and  $h \in C^{1,2}(\mathbb{R}_+ \times \bar{\Theta})$  for which there exist a constant  $C$  such that

$$H.5 \left\{ \begin{array}{l} (i) \quad l(x) \leq C(1 + |x|^p), \\ (ii) \quad h(t, x) \leq C(1 + |x|^p), \\ (iii) \quad h(0, x) \leq l(x). \end{array} \right.$$

where  $p$  is obtained in (H4).

For each fixed  $(t, x) \in \mathbb{R}_+ \times \bar{\Theta}$ , we consider  $(Y_s^\varepsilon, Z_s^\varepsilon, K_s^\varepsilon)_{0 \leq s \leq t}$ , the unique solution of reflected generalized BSDE

$$\left\{ \begin{array}{l} Y_s^\varepsilon = l(X_t^\varepsilon) + \int_s^t f(X_r^\varepsilon, Y_r^\varepsilon) dr + \int_s^t g(X_r^\varepsilon, Y_r^\varepsilon) dG_r^\varepsilon + K_t^\varepsilon - K_s^\varepsilon - \int_s^t Z_r^\varepsilon dM_r^{X^\varepsilon}, \\ Y_s^\varepsilon \geq h(s, X_s^\varepsilon), \\ K^\varepsilon \text{ is nondecreasing such that } K_0^\varepsilon = 0 \text{ and } \int_0^t (Y_s^\varepsilon - h(s, X_s^\varepsilon)) dK_s^\varepsilon = 0. \end{array} \right. \quad (3.1)$$

Let us note that existence and uniqueness of a solution to (3.1) is proved in [17].

Furthermore, we also consider  $(Y_s, Z_s, K_s)_{0 \leq s \leq t}$  the unique solution of the reflected BSDE

$$\left\{ \begin{array}{l} Y_s = l(X_t) + \int_s^t f(X_r, Y_r) dr + \int_s^t g(X_r, Y_r) dG_r - \int_s^t Z_r dM_r^X + K_t - K_s \\ Y_s \geq h(s, X_s) \\ K \text{ is nondecreasing such that } K_0 = 0 \text{ and } \int_0^t (Y_s - h(s, X_s)) dK_s = 0. \end{array} \right. \quad (3.2)$$

Throughout the rest of this paper, we put:

$$M_s^\varepsilon = \int_0^s Z_r^\varepsilon dM_r^{X^\varepsilon}, \quad M_s = \int_0^s Z_r dM_r^X; \quad 0 \leq s \leq t$$

and we consider  $(Y, M, K)$  (resp.,  $(Y^\varepsilon, M^\varepsilon, K^\varepsilon)$ ) as a random element of the space  $D([0, t]; \mathbb{R}) \times D([0, t]; \mathbb{R}) \times C([0, t]; \mathbb{R}_+)$ .  $D([0, t]; \mathbb{R})$  denotes the Skorohod space (the space of "cadlag" functions), endowed with the so-called  $S$ -topology of Jakubowski [8] and  $C([0, t]; \mathbb{R}_+)$  the space of functions of  $[0, t]$  with values in  $\mathbb{R}^d$  equipped with the topology of uniform convergence.

## 3.2 Weak convergence

The goal of this section is to prove the following result

**Theorem 3.1** *Under conditions **(H1)-(H4)** the family  $(Y^\varepsilon, M^\varepsilon, K^\varepsilon)_{\varepsilon>0}$  of processes converge in law to  $(Y, M, K)$  on the space  $D([0, t], \mathbb{R}) \times D([0, t], \mathbb{R}) \times C([0, t]; \mathbb{R}_+)$ . as  $\varepsilon \rightarrow 0$ . Moreover,*

$$\lim_{\varepsilon \rightarrow 0} Y_0^\varepsilon = Y_0 \text{ in } \mathbb{R}.$$

The proof of this theorem relies on four lemmas. The first one is a simplified version of Theorem 4.2 in (Billingsley [2], p. 25) so we omit the proof. We note also that the convergence is taken in law.

**Lemma 3.1** *Let  $(U^\varepsilon)_{\varepsilon>0}$  be a family of random variables defined on the same probability space. For each  $\varepsilon \geq 0$ , we assume the existence of a family random variables  $(U^{\varepsilon,n})_{n \leq 1}$ , such that:*

- (i) *For each  $n \geq 1$ ,  $U^{\varepsilon,n} \Rightarrow U^{0,n}$  as  $\varepsilon$  goes to zero and*
- (ii)  *$U^{0,n} \Rightarrow U^0$  as  $n \rightarrow +\infty$ .*
- (iii) *We suppose further that  $U^{\varepsilon,n} \Rightarrow U^\varepsilon$  as  $n \rightarrow +\infty$ , uniformly in  $\varepsilon$*

*Then,  $U^\varepsilon$  converges in law to  $U^0$ .*

For the last third lemmas, let us introduce the penalization method to construct a sequence of GBSDE. For each  $n \in \mathbb{N}^*$ , we set

$$f_n(x, y) = f(x, y) + n(y - h(s, x))^-$$

Thanks to the result of Pardoux and Zhang [16], for each  $n \in \mathbb{N}^*$ , there exists a unique pair of  $\mathcal{F}_t$ -progressively measurable processes  $(Y^{\varepsilon,n}, Z^{\varepsilon,n})$  with values in  $\mathbb{R} \times \mathbb{R}^d$  satisfying

$$\mathbb{E} \left[ \sup_{0 \leq s \leq t} |Y^{\varepsilon,n}|^2 + \int_0^t |Z_s^{\varepsilon,n}|^2 ds \right] < \infty$$

and

$$\begin{aligned} Y_s^{\varepsilon,n} &= l(X_t^\varepsilon) + \int_s^t f_n(X_r^\varepsilon, Y_r^{\varepsilon,n}) dr + \int_s^t g(X_r^\varepsilon, Y_r^{\varepsilon,n}) dG_r^\varepsilon \\ &\quad - (M_t^{\varepsilon,n} - M_s^{\varepsilon,n}), \end{aligned} \tag{3.3}$$

where

$$M_s^{\varepsilon,n} = \int_0^s Z_r^{\varepsilon,n} dM_r^{X^\varepsilon}.$$

From now on,  $C$  is a generic constant that may vary from line to another.

**Lemma 3.2** *Assume **(H1)-(H4)**. Then for any  $n$ , there exists the unique process  $(Y^{0,n}, M^{0,n})$  such that the family of processes  $(Y^{\varepsilon,n}, M^{\varepsilon,n})_{\varepsilon>0}$  converges to it in  $D([0, T], \mathbb{R}) \times D([0, T], \mathbb{R})$ , as  $\varepsilon$  goes to 0.*

**Proof. Step 1. A priori estimate uniformly in  $\varepsilon$  and  $n$**

Applying Itô's formula to the function  $|Y_t^{\varepsilon,n}|^2$ , we get

$$\begin{aligned} |Y_s^{\varepsilon,n}|^2 + \int_s^t |Z_r^{\varepsilon,n}|^2 dr &= |l(X_t^{\varepsilon})|^2 + 2 \int_s^t Y_r^{\varepsilon,n} f(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) dr \\ &\quad + 2 \int_s^t Y_r^{\varepsilon,n} g(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) dG_r^{\varepsilon} \\ &\quad - 2 \int_s^t Y_r^{\varepsilon,n} Z_r^{\varepsilon,n} dM_r^{X^{\varepsilon}} + 2 \int_s^t Y_r^{\varepsilon,n} dK_r^{\varepsilon,n}, \end{aligned} \quad (3.4)$$

where

$$K_s^{\varepsilon,n} = n \int_0^s [Y_r^{\varepsilon,n} - h(r, X_r^{\varepsilon,n})]^- dr \quad (3.5)$$

By using assumptions **(H3)** and **(H4)**, the inequality  $\int_s^t Y_r^{\varepsilon,n} dK_r^{\varepsilon,n} \leq \int_s^t h(r, X_r^{\varepsilon}) dK_r^{\varepsilon,n}$  and Young's inequality, we get for every  $\gamma > 0$

$$\begin{aligned} \mathbb{E} |Y_s^{\varepsilon,n}|^2 + \int_s^t |Z_r^{\varepsilon,n}|^2 dr &\leq (1 + C\mathbb{E}|X_t^{\varepsilon}|^{2p}) + \mathbb{E} \int_s^t \{(2\mu + 1)|Y_r^{\varepsilon,n}|^2 + C^2|X_r^{\varepsilon}|^{2q}\} dr \\ &\quad + \mathbb{E} \int_s^t \left\{ (2\beta + \gamma) |Y_r^{\varepsilon,n}|^2 + \frac{2C^2}{\gamma} |X_r^{\varepsilon}|^{2q} \right\} dG_r^{\varepsilon} \\ &\quad + \mathbb{E} \int_s^t h(r, X_r^{\varepsilon}) dK_r^{\varepsilon,n}. \end{aligned}$$

By virtue of assumption (h.i), Young's equality and choosing  $\gamma = |\beta|$  we obtain for every  $\delta > 0$

$$\begin{aligned} \mathbb{E} \left( |Y_s^{\varepsilon,n}|^2 + \int_s^t |Z_r^{\varepsilon,n}|^2 dr + |\beta| \int_s^t |Y_r^{\varepsilon,n}|^2 dG_r^{\varepsilon} \right) &\leq C(1 + \mathbb{E} \int_s^t |Y_r^{\varepsilon,n}|^2 dr) \\ &\quad + \frac{1}{\delta} \mathbb{E} \left( \sup_{0 \leq s \leq t} |h(s, X_s^{\varepsilon})|^2 \right) \\ &\quad + \delta \mathbb{E} (K_t^{\varepsilon,n} - K_s^{\varepsilon,n})^2. \end{aligned}$$

But

$$\begin{aligned} \mathbb{E} (K_t^{\varepsilon,n} - K_s^{\varepsilon,n})^2 &\leq C\mathbb{E} \left( |Y_s^{\varepsilon,n}|^2 + |l(X_t^{\varepsilon})|^2 + \int_s^t (|X_r^{\varepsilon}|^{2q} + |Y_r^{\varepsilon,n}|^2) dr \right) \\ &\quad + C\mathbb{E} \int_s^t (|X_r^{\varepsilon}|^{2q} + |Y_r^{\varepsilon,n}|^2) dG_r^{\varepsilon}. \end{aligned}$$

So, if  $\delta = \inf \left\{ \frac{1}{2C}, \frac{|\beta|}{2C} \right\}$ ,

$$\mathbb{E}|Y_s^{\varepsilon,n}|^2 + \mathbb{E} \int_s^t |Z_r^{\varepsilon,n}|^2 dr + \mathbb{E} \int_s^t |Y_r^{\varepsilon,n}|^2 dG_r^{\varepsilon} \leq C \left( 1 + \mathbb{E} \int_s^t |Y_r^{\varepsilon,n}|^2 dr \right)$$

and it follows from the Gronwall's lemma that

$$\mathbb{E} \left( |Y_s^{\varepsilon,n}|^2 + \int_s^t |Z_r^{\varepsilon,n}|^2 dr + \int_s^t |Y_r^{\varepsilon,n}|^2 dG_r^{\varepsilon} + |K_t^{\varepsilon,n}|^2 \right) \leq C, \quad 0 \leq s \leq t, \quad \forall n \in \mathbb{N}^*, \forall \varepsilon > 0.$$

Finally, Burkholder-Davis-Gundy inequality applying to (3.4) gives the following:

$$\sup_{n \in \mathbb{N}^*} \sup_{\varepsilon > 0} \mathbb{E} \left( \sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n}|^2 + \int_0^t |Z_r^{\varepsilon,n}|^2 dr + \int_0^t |Y_r^{\varepsilon,n}|^2 dG_r^{\varepsilon} + |K_t^{\varepsilon,n}|^2 \right) \leq C.$$

### Step 2: Tightness

We adopt the point of view of the  $S$ -topology of Jakubovski [8]. In fact we define the conditional variation of the process  $Y^{\varepsilon,n}$  on the interval  $[0, t]$  as the quantity

$$CV_t(Y^{\varepsilon,n}) = \sup_{\pi} \mathbb{E} \left( \sum_i |\mathbb{E}(Y_{t_{i+1}}^{\varepsilon,n} - Y_{t_i}^{\varepsilon,n} / \mathcal{F}_{t_i}^{\varepsilon})| \right)$$

where the supremum is taken over all partitions  $\pi : 0 = t_0 < t_1 < \dots < t_m = t$  of the interval  $[0, t]$ .

Clearly

$$CV_t(Y^{\varepsilon,n}) \leq \mathbb{E} \left( \int_0^t |f(X_r^{\varepsilon}, Y_r^{\varepsilon,n})| dr + \int_0^t |g(X_r^{\varepsilon}, Y_r^{\varepsilon,n})| dG_r^{\varepsilon} + |K_t^{\varepsilon,n}| \right),$$

and it follows from step 1 and (H3) that

$$\sup_{\varepsilon} \left( CV_t(Y^{\varepsilon,n}) + \mathbb{E} \left( \sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n}| + \sup_{0 \leq s \leq t} |M_s^{\varepsilon,n}| \right) + |K_t^{\varepsilon,n}|^2 \right) < +\infty.$$

Let us denote

$$H_s^{\varepsilon,n} = \int_0^s (g(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) + n[Y_r^{\varepsilon,n} - h(r, X_r^{\varepsilon})]^-) dG_r^{\varepsilon}.$$

By virtue of step 1 and (H3), we have

$$\sup_{\varepsilon} \left( CV_t(H^{\varepsilon,n}) + \mathbb{E} \left( \sup_{0 \leq s \leq t} |H_s^{\varepsilon,n}| \right) \right) < +\infty.$$

Hence the sequence  $\{(Y_s^{\varepsilon,n}, H_s^{\varepsilon,n}, M_s^{\varepsilon,n}); 0 \leq s \leq t\}$  satisfy Jakubowski tightness criterion for quasi-martingales under  $\mathbb{P}$  (see [8]), which is the same criterion as the one of Meyer-Zheng (see [11]).

**Step 3: Passage to the limit and it identification.**

After extracting a subsequence which we omit by abuse of notation, it follows by Jakubowski tightness criterion that  $(Y^{\varepsilon,n}, H^{\varepsilon,n}, M^{\varepsilon,n})$  converges in the weak sense to  $(Y^n, H^n, M^n)$  as  $\varepsilon$  goes to 0 in  $(D([0, t], \mathbb{R}))^3$  endowed with the product of  $S$ -topology.

Since  $f$  is continuous, the mapping  $(x, y) \mapsto \int_s^t (f(x(r), y(r)) + n[y(r) - h(r, x(r))]^-) dr$  is continuous from  $\mathcal{C}([0, t], \mathbb{R}^d) \times D([0, t], \mathbb{R})$ , equipped respectively by topology of uniform norm and the  $S$ -topology. Furthermore, it follows (see Boufoussi et al.[3]) that for all  $0 \leq s \leq t$

$$H_s^n = \int_0^s g(X_r, Y_r^{0,n}) dG_r$$

and

$$\begin{aligned} Y_s^{0,n} &= l(X_t) + \int_s^t f(X_r, Y_r^{0,n}) dr + \int_s^t g(X_r, Y_r^{0,n}) dG_r \\ &\quad + n \int_s^t (Y_r^{0,n} - h(r, X_r))^- dr + M_t^{0,n} - M_s^{0,n}. \end{aligned}$$

By using a similar argument as in Pardoux [15], one can prove that  $M^{0,n}$  and  $M^X$  are  $\mathcal{F}_s^{X,Y^n}$ -martingales. Let  $\{(\bar{Y}_s^n, \bar{U}_s^n), 0 \leq s \leq t\}$  denote the unique solution of the generalized BSDE

$$\begin{aligned} \bar{Y}_s^n &= l(X_t) + \int_s^t f(X_r, \bar{Y}_r^n) dr + \int_s^t g(X_r, \bar{Y}_r^n) dG_r \\ &\quad + n \int_s^t (\bar{Y}_r^n - h(r, X_r))^- dr + \bar{M}_t^n - \bar{M}_s^n, \end{aligned}$$

which satisfies  $\mathbb{E} \int_s^t \bar{U}_r^n d\langle M_r^X \rangle (\bar{U}_r^n)^* < +\infty$ . Set also  $\bar{M}_s^n = \int_0^s \bar{U}_r^n dM_r^X$ . Since  $\bar{Y}^n$ ,  $\bar{U}^n$  are  $\mathcal{F}_s^X$ -adapted, and  $M_s^X$ -the martingale part of  $X$  is a  $\mathcal{F}_s^{X,Y^n}$ -martingale, so is  $\bar{M}^n$ . It follows

from Itô's formula for possibly discontinuous semi- martingales that

$$\begin{aligned}
& \mathbb{E} |\bar{Y}_s^n - Y_s^n|^2 + \mathbb{E}[M^n - \tilde{M}^n]_t - \mathbb{E}[M^n - \tilde{M}^n]_s \\
= & 2\mathbb{E} \int_s^t (\bar{Y}_r^n - Y_r^n)(f(X_r, Y_r^n) - f(X_r, \bar{Y}_r^n))dr \\
& + 2\mathbb{E} \int_s^t (\bar{Y}_r^n - Y_r^n)(g(X_r, Y_r^n) - g(X_r, \bar{Y}_r^n))dG_r \\
& + 2\mathbb{E} \int_s^t (\bar{Y}_r^n - Y_r^n)(d\bar{K}^n - dK_r^n) \\
\leq & (2\mu + 1) \mathbb{E} \int_s^t |\bar{Y}_r^n - Y_r^n|^2 dr + 2\beta \mathbb{E} \int_s^t |\bar{Y}_r^n - Y_r^n|^2 dG_r \\
\leq & C \mathbb{E} \int_s^t |\bar{Y}_r^n - Y_r^n|^2 dr.
\end{aligned}$$

(We use the fact that  $\beta < 0$ ).

Hence, from Gronwall lemma  $\bar{Y}_s^n = Y_s^n$  and  $M_s^n = \tilde{M}_s^n$ ,  $0 \leq s \leq t$ . ■

**Lemma 3.3** *Assume **(H1)-(H4)**. Then the sequence of processes  $(Y^{0,n}, M^{0,n}, K^{0,n})_{n \geq 1}$  converges in law to  $(Y, M, K)$  in  $D([0, t], \mathbb{R}) \times D([0, t], \mathbb{R}) \times C([0, t], \mathbb{R}_+)$  endowed with the previous topology.*

**Proof.** By the same computation of step 1 and step 2 in the proof of Lemma 3.2, we get

$$\sup_n \mathbb{E} \left( \sup_{0 \leq s \leq t} |Y_s^{0,n}|^2 + \int_0^t |Z_r^{0,n}|^2 dr + \int_0^t |Y_r^{0,n}|^2 dG_r + |K_t^{0,n}|^2 \right) < +\infty$$

and

$$\sup_\varepsilon \left( CV_t(Y^{0,n}) + CV_t(K^{0,n}) + \mathbb{E} \left( \sup_{0 \leq s \leq t} |M_s^{0,n}|^2 \right) \right) < +\infty.$$

Further  $(Y^{n,0}, K^{0,n}, M^{0,n})$  satisfy Jakubowski tightness criterion for quasi-martingales under  $\mathbb{P}$  and by the similar argument as in Lemma 3.2,  $(Y^{n,0}, F^{0,n}, M^{0,n})$ , converges in law to  $(\hat{Y}, \hat{M}, \hat{K})$  as  $n$  goes to  $\infty$  on  $D([0, t], \mathbb{R}^2) \times C([0, t], \mathbb{R}_+)$  endowed with the previous topology. Finally by the uniqueness of the solution of the reflected generalized BSDE (3.2), it follows that  $(\hat{Y}, \hat{M}, \hat{K}) = (Y, M, K)$  which ends the proof. ■

**Lemma 3.4** *Assume **(H1)-(H4)**, the family of processes  $(Y^{\varepsilon,n}, M^{\varepsilon,n}, K^{\varepsilon,n})_{n \geq 1}$  converge uniformly in  $\varepsilon \in (0, 1]$  in probability to the family of processes  $(Y^\varepsilon, M^\varepsilon, K^\varepsilon)_{\varepsilon \in [0, 1]}$*

**Proof. Step 1:** In view of step 1 of the proof of Lemma 3.2

$$\sup_{\varepsilon} \sup_n \mathbb{E} \left( \sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n}|^2 + \int_0^t |Z_r^{\varepsilon,n}|^2 dr + \int_0^t |Y_r^{\varepsilon,n}|^2 dG_r^{\varepsilon} + |K_t^{\varepsilon,n}|^2 \right) < +\infty. \quad (3.6)$$

**Step 2:** There exists an  $\mathcal{F}_s$ -adapted process  $(\bar{Y}_s^{\varepsilon})_{0 \leq s \leq t}$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \int_0^t |Y_s^{\varepsilon,n} - \bar{Y}_s^{\varepsilon}|^2 ds = 0.$$

Notice that, for all  $n \geq 1$  and  $(s, x, y, z) \in [0, t] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$ ,

$$f_n(s, x, y, z) \leq f_{n+1}(s, x, y, z).$$

Therefore, by the comparison theorem for generalized BSDE [16], we have  $Y_s^{\varepsilon,n} \leq Y_s^{\varepsilon,n+1}$ , a.s. for all  $0 \leq s \leq t$ . Hence, for all  $0 \leq s \leq t$

$$Y_s^{\varepsilon,n} \nearrow \bar{Y}_s^{\varepsilon}, \quad a.s.$$

In view of inequality (3.6) and Fatou's lemma,

$$\mathbb{E} \left( \sup_{0 \leq s \leq t} |\bar{Y}_s^{\varepsilon}|^2 \right) < C.$$

It then follows by the Lebesgue convergence theorem that

$$\lim_n \mathbb{E} \int_0^t |Y_r^{\varepsilon,n} - \bar{Y}_r^{\varepsilon}|^2 dr = 0.$$

**Step 3:**  $\lim_{n \rightarrow \infty} \mathbb{E}(\sup_{0 \leq s \leq t} |(Y_s^{\varepsilon,n} - h(s, X_s^{\varepsilon}))^-|^2) = 0$  a.s.

Let  $\{(\tilde{Y}_s^{\varepsilon,n}, \tilde{Z}_s^{\varepsilon,n}), 0 \leq s \leq t\}$  be the unique solution of BSDE

$$\begin{aligned} \tilde{Y}_s^{\varepsilon,n} &= l(X_t^{\varepsilon}) + \int_s^t f(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) dr + \int_s^t g(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) dG_r^{\varepsilon} \\ &\quad + n \int_s^t (h(r, X_r^{\varepsilon}) - \tilde{Y}_r^{\varepsilon,n}) dr - \int_s^t \tilde{Z}_r^{\varepsilon,n} dM_r^{X^{\varepsilon}} \end{aligned}$$

By using the comparison theorem, it follows for all  $0 \leq s \leq t$  and  $n \geq 1$  that  $\tilde{Y}_s^{\varepsilon,n} \leq Y_s^{\varepsilon,n}$ , a.s.

Now let  $\nu$  be an  $\mathcal{F}_t$ -stopping time such that  $0 \leq \nu \leq t$ . Then, applying Itô's formula to  $\tilde{Y}_s^{\varepsilon,n} e^{n(t-\nu)}$ , we have

$$\begin{aligned} \tilde{Y}_{\nu}^{\varepsilon,n} &= \mathbb{E}^{\mathcal{F}_{\nu}}[e^{-n(t-\nu)} l(X_t^{\varepsilon}) + \int_{\nu}^t e^{-n(s-\nu)} f(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) dr \\ &\quad + \int_{\nu}^t e^{-n(r-\nu)} g(X_r^{\varepsilon}, Y_r^{\varepsilon,n}) dG_r^{\varepsilon} + n \int_{\nu}^t e^{-n(r-\nu)} h(r, X_r^{\varepsilon}) dr]. \end{aligned}$$

It is easily seen that

$$e^{-n(t-\nu)}l(X_t^\varepsilon) + n \int_\nu^t e^{-n(r-\nu)}h(r, X_r^\varepsilon) dr \rightarrow l(X_t^\varepsilon) \mathbf{1}_{\{\nu=t\}} + h(t, X_\nu^\varepsilon) \mathbf{1}_{\{\nu< t\}}$$

a.s. and in  $L^2(\Omega)$ , and the conditional expectation converges also in  $L^2(\Omega)$ . Moreover,

$$\begin{aligned} \left| \int_\nu^t e^{-n(r-\nu)} f(X_r^\varepsilon, Y_r^{\varepsilon,n}) dr \right| &\leq \frac{1}{\sqrt{n}} \left( \int_0^t |f(X_r^\varepsilon, Y_r^{\varepsilon,n})|^2 dr \right)^{1/2} \\ \left| \int_\nu^t e^{-n(r-\nu)} g(X_r^\varepsilon, Y_r^{\varepsilon,n}) dG_r^\varepsilon \right| &\leq \frac{1}{\sqrt{n}} \left( \int_0^t |g(X_r^\varepsilon, Y_r^{\varepsilon,n})|^2 dG_r^\varepsilon \right)^{1/2}, \end{aligned}$$

and

$$\mathbb{E}^{\mathcal{F}_\nu} \left\{ \int_\nu^t e^{-n(r-\nu)} f(X_r^\varepsilon, Y_r^{\varepsilon,n}) dr + \int_\nu^t e^{-n(r-\nu)} g(X_r^\varepsilon, Y_r^{\varepsilon,n}) dG_r^\varepsilon \right\} \rightarrow 0$$

in  $L^2(\Omega)$  as  $n \rightarrow \infty$ . Consequently,

$$\tilde{Y}_\nu^{\varepsilon,n} \longrightarrow l(X_t^\varepsilon) \mathbf{1}_{\{\nu=t\}} + h(\nu, X_\nu^\varepsilon) \mathbf{1}_{\{\nu< t\}} \text{ in } L^2(\Omega) \text{ as } n \rightarrow \infty.$$

Therefore,  $\bar{Y}_\nu \geq h(\nu, X_\nu)$  a.s.

From this and the section theorem (see Dellacherie et Meyer [4] page 220), the previous inequality hold for all  $s \in [0, t]$  a.s. Then

$$(Y_s^n - h(s, X_s))^- \searrow 0, \quad 0 \leq s \leq t, \text{ a.s.},$$

and from Dini's theorem the convergence is uniform in  $t$ . Since  $(Y_s^n - h(s, X_s))^- \leq (h(s, X_s) - Y_s^0)^+ \leq |h(s, X_s)| + |\bar{Y}_s^0|$ , the result follows by dominated convergence.

**Step 4:**  $\lim_{n \rightarrow \infty} \mathbb{E} \left( \sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n} - \bar{Y}_s|^2 \right) = 0$  and there exists  $\mathcal{F}_t$ -adapted processes  $(\bar{Z}_s)_{0 \leq s \leq t}$  and  $(\bar{K}_s)_{0 \leq s \leq t}$  such that

$$\lim_{n \rightarrow \infty} \mathbb{E} \left( \int_0^T |Z_r^{\varepsilon,n} - \bar{Z}_r|^2 + \sup_{0 \leq s \leq t} |K_s^{\varepsilon,n} - \bar{K}_s|^2 \right) = 0.$$

Indeed, by Itô's formula and taking expectation in both sides, we have for all  $n \geq m \geq 1$  and  $0 \leq s \leq t$ ,

$$\begin{aligned} &|Y_s^{\varepsilon,n} - Y_s^{\varepsilon,m}|^2 + \int_s^t |Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m}|^2 dr + 2|\beta| \int_s^t |Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m}|^2 dG_r^\varepsilon \\ &\leq 2\mu \int_s^t |Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m}|^2 dr + 2 \int_s^t (Y_r^{\varepsilon,n} - h(t, X_r^\varepsilon))^- dK_r^{\varepsilon,n} \\ &\quad + 2 \int_s^t (Y_r^{\varepsilon,m} - h(r, X_r^\varepsilon))^- dK_r^{\varepsilon,m} - \int_s^t (Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m}) (Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m}) dM_r^{X^\varepsilon}. \end{aligned} \quad (3.7)$$

Therefore

$$\begin{aligned}\mathbb{E} \int_s^t |Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m}|^2 dr &\leq C\mathbb{E} \int_s^t |Y_r^{\varepsilon,n} - Y_r^{\varepsilon,m}|^2 dr + 2\mathbb{E} \int_s^t (Y_r^{\varepsilon,n} - h(r, X_r^{\varepsilon}))^- dK_r^{\varepsilon,n} \\ &\quad + 2\mathbb{E} \int_s^t (Y_r^{\varepsilon,m} - h(r, X_r^{\varepsilon}))^- dK_r^{\varepsilon,m}.\end{aligned}$$

Furthermore, Hölder's inequality provide

$$\mathbb{E} \int_s^t (Y_r^{\varepsilon,n} - h(r, X_r^{\varepsilon}))^- dK_r^{\varepsilon,n} \leq \left[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |(Y_s^{\varepsilon,n} - h(s, X_s))^ -|^2 \right) \right]^{1/2} [\mathbb{E} |K_t^m|^2]^{1/2} \quad (3.8)$$

and

$$\mathbb{E} \int_s^t (Y_r^{\varepsilon,p} - h(r, X_r^{\varepsilon}))^- dK_r^{\varepsilon,n} \leq \left[ \mathbb{E} \left( \sup_{0 \leq s \leq t} |(Y_s^{\varepsilon,m} - h(s, X_s))^ -|^2 \right) \right]^{1/2} [\mathbb{E} |K_t^n|^2]^{1/2}. \quad (3.9)$$

By using the result of Step 1 and Step 2, the right hands sides of the above inequalities (3.8) and (3.9) go to zero as  $m, n \rightarrow \infty$ . It follows that

$$\mathbb{E} \int_0^t |Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m}|^2 dr \rightarrow 0$$

as  $n, m \rightarrow \infty$ , and there exists a process  $\bar{Z}^{\varepsilon} \in H^{2,d}(0, t)$  such that

$$\mathbb{E} \int_0^t |Z_r^{\varepsilon,n} - \bar{Z}_r^{\varepsilon}|^2 dr \rightarrow 0, \text{ as } n \rightarrow \infty.$$

Then, by the Burkholder-Davis-Gundy inequality, we get

$$\begin{aligned}\mathbb{E} \left( \sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n} - Y_s^{\varepsilon,m}|^2 \right) &\leq 4\mathbb{E} \int_0^t (Y_s^n - S_s)^- dK_s^m + 4\mathbb{E} \int_0^t (Y_s^m - S_s)^- dK_s^n \\ &\quad + C\mathbb{E} \int_0^t |Z_r^{\varepsilon,n} - Z_r^{\varepsilon,m}| ds.\end{aligned}$$

It follows that  $\mathbb{E}(\sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n} - Y_s^{\varepsilon,m}|^2) \rightarrow 0$  as  $n, m \rightarrow \infty$ . Therefore,

$$\mathbb{E} \left\{ \sup_{0 \leq s \leq t} |Y_s^{\varepsilon,n} - \bar{Y}_s^{\varepsilon}|^2 \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

We deduce that

$$\mathbb{E} \left( \sup_{0 \leq s \leq t} |K_s^{\varepsilon,n} - K_s^{\varepsilon,m}|^2 \right) \rightarrow 0 \text{ as } n, m \rightarrow \infty.$$

Hence there exists a process  $(\bar{K}_s^\varepsilon)_{0 \leq s \leq t} \in A^2(0, t)$  such that

$$\mathbb{E} \left( \sup_{0 \leq s \leq t} |K_s^{\varepsilon, n} - \bar{K}_s^\varepsilon|^2 \right) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

**Step 5:** The limiting process  $(\bar{Y}^\varepsilon, \bar{Z}^\varepsilon, \bar{K}^\varepsilon)$  is the solution of the reflected generalized BSDE  $(\xi, f, g, h)$ .

Obviously the process  $(\bar{Y}_s^\varepsilon, \bar{Z}_s^\varepsilon, \bar{K}_s^\varepsilon)_{0 \leq s \leq t}$  satisfies

$$\bar{Y}_s^\varepsilon = \xi + \int_s^t f(r, X_r^\varepsilon, \bar{Y}_r^\varepsilon, \bar{Z}_r^\varepsilon) ds + \int_s^t g(r, X_r^\varepsilon, \bar{Y}_r^\varepsilon) dG_r^\varepsilon + \bar{K}_t^\varepsilon - \bar{K}_s^\varepsilon - \int_s^t \bar{Z}_r^\varepsilon dM_r^{X^\varepsilon}, \quad 0 \leq s \leq t.$$

Since  $(Y^{\varepsilon, n}, K^{\varepsilon, n})$  tends to  $(\bar{Y}^\varepsilon, \bar{K}^\varepsilon)$  in probability uniformly in  $t$ , the measure  $dK^{\varepsilon, n}$  converges to  $d\bar{K}^\varepsilon$  weakly in probability, so that  $\int_0^t (Y_r^{\varepsilon, n} - h(r, X_r^\varepsilon)) dK_r^{\varepsilon, n} \rightarrow \int_0^t (\bar{Y}_r^\varepsilon - h(r, X_r^\varepsilon)) d\bar{K}_r^\varepsilon$  as  $n \rightarrow \infty$ . Obviously,  $\int_0^t (\bar{Y}_r^\varepsilon - h(r, X_r^\varepsilon)) d\bar{K}_r^\varepsilon \geq 0$ , while, on the other hand, for all  $n \geq 0$ ,  $\int_0^t (Y_r^{\varepsilon, n} - h(r, X_r^\varepsilon)) dK_r^{\varepsilon, n} \leq 0$ .

Hence,

$$\int_0^t (\bar{Y}_s^\varepsilon - h(s, X_s^\varepsilon)) d\bar{K}_s^\varepsilon = 0. \quad \text{a.s.}$$

Consequently,  $(\bar{Y}_s^\varepsilon, \bar{Z}_s^\varepsilon, \bar{K}_s^\varepsilon)_{0 \leq s \leq t}$  is the solution of the reflected generalized BSDE  $(\xi, f, g, h)$  and uniqueness of this equation allows that  $\bar{Y}^\varepsilon = Y^\varepsilon$ ,  $\bar{Z}^\varepsilon = Z^\varepsilon$  and  $\bar{K}^\varepsilon = K^\varepsilon$  which ends the proof.  $\blacksquare$

### Proof of Theorem 3.1

Put Theorem 2.1, Lemma 3.2, Lemma 3.3, Lemma 3.4 and the Proposition 3.1 together prove that  $(Y^\varepsilon, M^\varepsilon, K^\varepsilon)$  converges in law to  $(Y, M, K)$  in  $D([0, T], \mathbb{R}) \times D([0, T], \mathbb{R}) \times C([0, T], \mathbb{R}_+)$ , where the first factor is equipped with the  $S$ -topology, and the two last factors with the topology of uniform convergence. In particular

$$Y_0^\varepsilon = \mathbb{E} \left\{ l(X_t^\varepsilon) + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t g(X_s^\varepsilon, Y_s^\varepsilon) dG_s^\varepsilon + K_t^\varepsilon \right\}.$$

converge to

$$Y_0 = \mathbb{E} \left\{ l(X_t) + \int_0^t f(X_s, Y_s) ds + \int_0^t g(X_s, Y_s) dG_s + K_t \right\}.$$

in  $\mathbb{R}$ . Indeed, since  $Y_0^\varepsilon$  is deterministic,  $Y_0^\varepsilon = \mathbb{E}(B_\varepsilon)$  where

$$B_\varepsilon = l(X_t^\varepsilon) + \int_0^t f(X_s^\varepsilon, Y_s^\varepsilon) ds + \int_0^t g(X_s^\varepsilon, Y_s^\varepsilon) dG_s^\varepsilon + K_t^\varepsilon.$$

In view of assumptions **(H1)-(H4)**, we have,

$$\sup_{\varepsilon} \mathbb{E}|B_{\varepsilon}|^2 \leq C.$$

Since  $B_{\varepsilon}$  converge in law as  $\varepsilon$  goes to 0, toward

$$B = l(X_t) + \int_0^t f(X_s, Y_s) ds + \int_0^t g(X_s, Y_s) dG_s + K_t,$$

the uniformly integrability of  $B_{\varepsilon}$  implies that

$$\lim_{\varepsilon \rightarrow 0} \mathbb{E}(B_{\varepsilon}) = \mathbb{E}\left(\lim_{\varepsilon \rightarrow 0} B^{\varepsilon}\right).$$

This mean that  $Y^{\varepsilon}$  converges to  $Y_0 = \mathbb{E}(B)$ .

■

## 4 Homogenization of the reflected viscosity solutions of scalar PDE with non linear Neumann boundary condition

The result of the above section permits to us to deduce weak convergence of a sequence  $(Y^{\varepsilon})$  of the solutions of reflected generalized BSDEs from weak convergence of the sequence  $(X^{\varepsilon})$ . In a sense, we deduce, from the probabilistic proof of the convergence of linear PDE with linear boundary condition due to Tanaka [18], a probabilistic proof of convergence of reflected semi-linear PDEs with non linear Neumann boundary condition.

For each  $x \in \bar{\Theta}$ , let  $\{(X_s^{\varepsilon,x}, G_s^{\varepsilon,x}), s \geq 0\}$  denote the solution of the SDE (2.3). Let  $l, f, g, h$  be as in the previous section. For each  $(t, x) \in \mathbb{R}_+ \times \bar{\Theta}$ , let

$$u^{\varepsilon}(t, x) = Y_0^{\varepsilon,x},$$

where  $Y^{\varepsilon}$  denote the solution of the reflected generalized BSDE (3.1) considered in the previous section (which of course depends on the starting point  $x$  of  $X^{\varepsilon}$ , and the final time  $t$  for the reflected generalized BSDE). The continuity of  $u^{\varepsilon}$  follows by the standard arguments (see [17]) from the continuity of the mapping  $x \mapsto X^{\varepsilon,x}$  in probability. One can show as in Ren and Xia [17] that  $u^{\varepsilon}$  is the viscosity solution of the semi-linear parabolic PDEs

$$\begin{cases} \min(u^{\varepsilon}(t, x) - h(t, x), \\ -\frac{\partial u^{\varepsilon}}{\partial t}(t, x) - L_x^{\varepsilon}u^{\varepsilon}(t, x) - f(x, u^{\varepsilon}(t, x)) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \Theta, \\ \Gamma^{\varepsilon}u^{\varepsilon}(t, x) + g(x, u^{\varepsilon}(t, x)) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \partial\Theta, \\ u^{\varepsilon}(0, x) = l(x), \quad x \in \bar{\Theta}, \end{cases} \quad (4.1)$$

where  $L_x^\varepsilon$  and  $\Gamma_x^\varepsilon$  are the operators defined by (2.1) and (2.2) respectively.

Let  $u$  be the solution of the homogenized PDE

$$\begin{cases} \min (u(t, x) - h(t, x), \\ -\frac{\partial u}{\partial t}(t, x) - L_x^0 u(t, x) - f(x, u(t, x)) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \Theta, \\ \Gamma u(t, x) + g(x, u(t, x)) = 0, \text{ if } (t, x) \in \mathbb{R}_+ \times \partial\Theta, \\ u(0, x) = l(x), \quad x \in \overline{\Theta}, \end{cases} \quad (4.2)$$

where  $L_x^0$  and  $\Gamma_x^0$  are defined in (2.4). Recall again the previous argument,  $u(t, x) = Y_0^x$  is the viscosity solution of (4.2), where  $Y^x$  denote the solution of the reflected generalized BSDE (3.2), which depend on the starting point of  $X^x$ .

**Theorem 4.1** *Under the assumptions **(H1)-(H4)**, for each  $(t, x) \in \mathbb{R}_+ \times \Theta$ , Then  $u^\varepsilon(t, x)$  converges to  $u(t, x)$  as  $\varepsilon$  goes to 0.*

**Proof.** Since  $u^\varepsilon(t, x) = Y_0^{\varepsilon, x}$  and  $u(t, x) = Y_0^x$ , the result follows from Theorem 3.1 ■

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